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Stochastic Integral Representation of Some Martingales

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Let $\{X_t\}$ be a continuous square integrable martingale. Denote its increasing (natural) process by $\{A_t\}$. Let S_t, T_t be the left and right inverses of A_t, respectively. Then for any square integrable martingale $\{Y_t\}$ defined on $\{X_t\}$, $Y_t = \int_0^t \psi_s \, dX_s$, $R_0 < t < S_\infty$ where $S_\infty = \lim_{t\to\infty} S_t$, $R_0 = \inf\{t: X_t \neq 0\}$ provided that $Y(T(t))$ is $\sigma(X(T(s)): s \leqslant t)$-measurable. All martingales are assumed to be zero at $t = 0$. Brownian motion and Poisson processes are considered also.

1. INTRODUCTION

It is well known [4, 6, 11, 16] that if $\{X_t\}$ is a martingale over a standard Brownian motion $\{B_t\}$ $X_t = EX_t + \int_0^t \phi_s \, dB_s$. Maisonneuve [12] gives an example which shows that the assertion is false if $\{B_t\}$ is replaced by a continuous square integrable martingale. Section 5 of this paper includes other examples.

These and other processes are defined on a probability space $(\Omega, \mathfrak{A}, P)$. Such a space will be assumed, and reference to it may not be explicit throughout.

In Section 4, we extend the forgoing representation to the case where the underlying process is a continuous martingale, whose associated increasing process does not depend on $\omega \in \Omega$. Our method is a combination of mathematical induction and Ito's formula for martingales [11, 15]. The method is illustrated best in the Brownian motion case which we treat in Section 3. Results of general character, based on random time change techniques, will be given in Section 5. The same techniques of Sections 3 and 4 may be used to handle discontinuous martingales. We treat the Poisson process as an example in Section 6.

If $\{Z_t\}$, $t \geqslant 0$ is a stochastic process, we write $\sigma(Z_s: s \leqslant t)$ for the σ-algebra generated by $(Z_s: s \leqslant t)$. We say that Z_t is continuous if it has continuous paths. In general, the following statements hold. $V_t \mathfrak{A}_t$ denotes the σ-algebra generated by $(\mathfrak{A}_s: s \geqslant 0)$. $1_A = 1$ on A, $=$ zero off A. Unexplained notation and terminology are found in [12].

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2. Stochastic Integrals

Literature on stochastic integration is overwhelmed by terminology. To dispel confusion, we explain in this section stochastic integration relative to right-continuous martingales. We shall specify (explicitly) the class of objects to be integrated. This class is fairly wide when one integrates relative to Brownian motion, but not as wide in general, and care must be given accordingly. No proof will be given. The reader may consult [11, 16].

Let $(\Omega, \mathfrak{A}, P)$ be a probability space, $\mathfrak{A}_t$, $t \geqslant 0$ an increasing right-continuous family of sub-σ-algebras of $\mathfrak{A}$. We assume that $\mathfrak{A}_0$ contains all P negligible sets. Let

$$\mathscr{M} = \{M \colon M = (M_t, \mathfrak{A}_t), \text{ right-continuous martingale, } M_0 = 0, \\ \text{and } \sup_t EM_t^2 < +\infty\}.$$

For $X \in \mathscr{M}$ we have Doob–Meyer [8, 13–15] decomposition.

$$X_t^2 = M_t + \langle X, X\rangle_t, \tag{I}$$

where $M = (M_t, \mathfrak{A}_t) \in \mathscr{M}$, and $\langle X, X\rangle_t$ is the unique (natural) increasing process.

Also, for $X, Y \in \mathscr{M}$, we have by (I):

$$(X_t + Y_t)^2 = M_t^{(1)} + \langle X + Y\rangle_t,$$
$$(X_t - Y)^2 = M_t^{(2)} + \langle X - Y\rangle_t,$$

Thus,

$$X_tY_t = M_t + \tfrac{1}{4}[\langle X + Y\rangle - \langle X - Y\rangle].$$

Write

$$\langle X, Y\rangle = \tfrac{1}{4}[\langle X + Y\rangle - \langle X - Y\rangle];$$

$\langle X, Y\rangle$ is the unique (natural) bounded variation process, such that $XY - \langle X, Y\rangle$ is a martingale. The reader is referred to [9] for properties and use of $\langle X, Y\rangle$.

To abbreviate our notation, we sometimes write A_t in place of $\langle X, X\rangle_t$. Let P denote the σ-algebra generated by the sets of the form $(s, t] \times A_s$, where $s < t$ and $A_s \in \mathfrak{A}_s$. P is called a predictable σ-algebra (other terms may be used) and is very well measurable. Intervals $(s, t]$ may be replaced by stochastic intervals $(S, T]$ (see [7, 17]).

$$L^2(P) = \Big\{f : f \colon [0, +\infty) \times \Omega \to R, \ P\text{-measurable}, f(t, \cdot) \text{ is } \mathfrak{A}_t\text{-adapted}, \\ \text{and } E\int_0^{+\infty} f^2(s, \omega)\, dA_s < +\infty\Big\}.$$

As usual R denotes the reals and adapted means measurable. A function of the form

$$f(t, \omega) = \sum_{k=1}^{n} g(s_k, \omega)\, 1_{(s_k, t_k]}(t),$$

where $s_1 < t_1 \leqslant s_2 < t_2 \leqslant \cdots$ and $g(s_k, \cdot)$ is $\mathfrak{A}_{s_k}$-measurable is clearly a P-measurable simple function. Put

$$\int_0^{+\infty} f(s, \omega)\, dX_s = \sum_{k=1}^{n} g(s_k, \omega)\, [X_{t_k} - X_{s_k}].$$

Replacing f by $f \cdot 1_{[0,t]}$ we define

$$\int_0^t f(s, \omega)\, dX_s = \int_0^{+\infty} f(s, \omega)\, 1_{(0,t]}(s)\, dX_s .$$

Some properties are: $\int_0^t f(s, \omega)\, dX_s$ is a martingale,

$$E \int_0^t f(s, \omega)\, dX_s = 0 \qquad \text{and} \qquad E\left[\int_0^t f(s, \omega)\, dX_s\right]^2 = E\left[\int_0^t f^2(s, \omega)\, dA_s\right]. \tag{II}$$

The simple functions in $L^2(P)$ are dense in $L^2(P)$, relative to the metric induced by (II). This fact enables us to define $\int_0^t f(s, \omega)\, dX_s$ for all $f \in L^2(P)$.

From now on except in rare circumstances we shall abbreviate $\int_0^t f(s, \omega)\, dX_s$ by $\int_0^t f_s\, dX_s$ or by $\int_0^t f\, dX$.

We must remark that $\int_0^t f_s\, dX_s$ is right-continuous (continuous) or rather can be defined as such, if $\{x_t\}$ is right-continuous (continuous). Also that:

$$\left\langle \int f_s\, dX_s, Z \right\rangle_t = \int_0^t f_s\, d\langle X, Z\rangle_s$$

for all $Z \in \mathscr{M}$.

Finally, one may relax the condition $\sup_t EX_t^2 < +\infty$ in the definition of $\mathscr{M}$ to $EX_t^2 < +\infty$ for every $t \geqslant 0$, with the corresponding condition $E[\int_0^t f_s^2\, dA_s] < +\infty$ for every $t \geqslant 0$, in the definition of $L^2(P)$.

Two elements $X, Y \in \mathscr{M}$ are called orthogonal, written $X \perp Y$ if $\langle X, Y\rangle = 0$.

A subset $\mathscr{N} \subset \mathscr{M}$ is called a stable subspace [11, 16] (i) if $X, Y \in \mathscr{N} \Rightarrow X + Y \in \mathscr{N}$ (ii) if $X \in \mathscr{N}$ then $\int_0^t f_s\, dX_s \in \mathscr{N}$ for all $f \in L^2(P)$, and (iii) if $\mathscr{N}$ is closed in the metric induced by $\sup_t EX_t^2 < +\infty$. It is easy to show that $\mathscr{N}(X) = \{\int_0^t f_s\, dX_s : f \in L^2(P)\}$, $X \in \mathscr{M}$, is a stable subspace.

If X, Y are elements of $\mathscr{M}$ then $Y = Y_1 + Y_2$ (uniquely), $Y_1 \in \mathscr{N}(X)$, $Y_2 \perp \mathscr{N}(X)$.

3. Brownian Motion Case

Let $\{B_t\}$, $t \geqslant 0$ be a standard Brownian motion with $B_0 = 0$. Let $\mathfrak{A}_t = \sigma(B_s: s \leqslant t)$ be the σ-algebra generated by $(B_s: s \leqslant t)$. It is assumed that $\mathfrak{A}_0$ contains all negligible sets, and that $\{\mathfrak{A}_t\}$, $t \geqslant 0$, is right-continuous.

Theorem 1. *Let $(Y_t, \mathfrak{A}_t)$ be a right-continuous martingale satisfying $\sup_t EY_t^2 < +\infty$ and $Y_0 = 0$. Then*

$$Y_t = \int_0^t f_s \, dB_s, \qquad f \in L^2(P).$$

$Y_0 = 0$ is no real restriction. It only simplifies the notation; i.e., if $Y_0 \neq 0$ we replace Y_t by $Y_t - Y_0$.

The proof will make use of the following simple lemma.

Lemma 1. *Let $(X_t, \mathscr{F}_t)$ and $(Y_t, \mathscr{F}_t)$ be two arbitrary martingales. If $EX_tY_t = 0$, for every $t \geqslant 0$ then $EX_sY_t = 0$ for all $s, t \geqslant 0$.*

Proof. $EX_sY_t = EE\{X_sY_t \mid \mathscr{F}_{s\wedge t}\} = 0$. Here $s \wedge t$ denotes the minimum of s and t.

Proof of Theorem 1. By Section 2 we decompose Y uniquely as $Y = Y^{(1)} + Y^{(2)}$, where $Y_t^{(1)} = \int_0^t f_s \, dB_s$ and $Y^{(2)} \perp \mathscr{N}(B)$. (Recall that $\mathscr{N}(B)$ is the stable space generated by $B = \{B_t\}$.) We will show $Y^{(2)} = 0$ by showing $Y_{t0}^{(2)} = 0$ for an arbitrary but a fixed t_0. Since for an arbitrary finite sequence of nonnegative numbers $t_1, t_2, \ldots, t_n$, $B_{t_1}B_{t_2} \cdots B_{t_n} \in L^2(B_s: s \geqslant 0)$, the set of linear combinations of functions of the form $B_{t_1}B_{t_2} \cdots B_{t_n}$, $n \geqslant 1$, is dense in $L^2(B_s: s \geqslant 0)$. Thus it suffices to show that $EY_{t0}^{(2)}B_{t_1}B_{t_2} \cdots B_{t_n} = 0$. By Lemma 1, $EY_{t0}^{(2)}B_t = 0$ for every $t \geqslant 0$. Assume that $EY_{t0}^{(2)}B_{t_1}B_{t_2} \cdots B_{t_n} = 0$ for all $0 \leqslant t_i < +\infty$, $i = 1, 2, \ldots, n$. Introduce the functions φ_i and φ_{ij}, defined as

$$\varphi_i(B_{t_1}B_{t_2} \cdots B_{t_k}) = B_{t_1}B_{t_2} \cdots B_{t_{i-1}}B_{t_{i+1}} \cdots B_{t_k},$$

$$\varphi_{ij}(B_{t_1}B_2 \cdots B_{t_k}) = B_{t_1}B_{t_2} \cdots B_{t_{i-1}}B_{t_{i+1}} \cdots B_{t_{j-1}}B_{t_{j+1}} \cdots B_{t_k},$$

i.e., φ_i knowcks out the ith factor and φ_{ij} knocks out ith and jth factors. By Ito's formula [11, 15], applied on $B_{t\wedge t_1}B_{t\wedge t_2} \cdots B_{t\wedge t_{n+1}}$, we have

$$B_{t\wedge t_1}B_{t\wedge t_2} \cdots B_{t\wedge t_{n+1}} = \sum_{i=1}^{n+1} \int_0^t \varphi_i \, dB_{s\wedge t_i} + \tfrac{1}{2} \sum_{i,j=1}^{n+1} \int_0^t \varphi_{ij} d\langle B_{s\wedge t_i}, B_{s\wedge t_j}\rangle.$$

Here φ_i, φ_{ij} are acting on $B_{s\wedge t_1}B_{s\wedge t_2} \cdots B_{s\wedge t_{n+1}}$.

Take $t \geqslant$ maximum of $t_1, \ldots, t_{n+1}$. The left-hand side is just $B_{t_1} B_{t_2} \cdots B_{t_{n+1}}$. By Lemma 1, $EY_{t0}^{(2)} \cdot \sum_{i=1}^{n+1} \int_0^{t_i} \varphi_i \, dB_{s \wedge t_i} = 0$. Since $\langle B_{s \wedge t_i}, B_{s \wedge t_j} \rangle$ equals s or zero, $EY_{t0}^{(2)} \cdot \frac{1}{2} \sum_{i,j=1}^{n+1} \int_0^t \varphi_{ij} \, d\langle B_{s \wedge t_i} B_{s \wedge t_j} \rangle = 0$, by Fubini's theorem and the induction hypothesis. The proof is complete.

4. An Extension

We continue with the notation and assumptions given in Section 2.

Theorem 4.1. *If there is an $X \in \mathscr{M}$, X is continuous, $\mathfrak{A}_t = \sigma(X_s \colon s \leqslant t)$, and A_t is independent of $\omega \in \Omega$, then $Y_t = \int_0^t \varphi_s \, dX_s$, for every $Y \in \mathscr{M}$. Here φ depends on Y.*

Proof. $E \mid X_t \mid^p < +\infty$, $0 < p < +\infty$ follows from Burkholder's inequality [2]. (See a neat proof of this in [5, pp. 276].)

Alternatively the assertion follows from the random time change technique; however, we take up this technique in Section 5. As before, $X_{t_1} \cdots X_{t_n} \in L^2(X_s \colon s \geqslant 0)$ by Holder's inequality. Again the set of linear combinations of functions of the form $X_{t_1} \cdots X_{t_n}$ is dense in $L^2(X_s \colon s \geqslant 0)$.

The rest of the proof is exactly as before, except for the change of B_t to X_t.

It might be useful to remark that $\exp\{B_t - (t/2)\}$ is a martingale, whose associated increasing process is not independent of $\omega \in \Omega$, yet every martingale defined on it is given by stochastic integral. To see this, one need only notice that $e^{B_t - (t/2)} = 1 + \int_0^t e^{B_s - (s/2)} \, dB_s$.

5. General Case

The main tool in this section is the so-called random time change [1, 10, 11]. To begin, let $X \in \mathscr{M}$, suppose X is continuous, and further, assume that $\mathfrak{A}_t = \sigma(X_s \colon s \leqslant t)$.

For $Y \in \mathscr{M}$, to be written as $Y_t = \int_0^t \varphi_s \, dX_s$, Y must be continuous, and its paths must be constants whenever paths of X are constants. More precisely, if C_z = the union of the intervals of constancy of a process Z, then we must have $C_x \subset C_y$.

Assisted by Examples 5.1 and 5.2 below we shall conclude that there are Y's $\in \mathscr{M}$ which do not satisfy the condition $C_x \subset C_y$. In other words, stochastic integral representation is not valid for square integrable martingales, in general. Maisonneuve [12], in his example, exploits the continuity aspect of the problem. To arrive at the examples, and proceed further, we must consider several measurability problems. An $\mathfrak{A}_t$-stopping time or more briefly a stopping T is an extended real-valued random variable such that $\{T \leqslant t\} \in \mathfrak{A}_t$ for every t. $\mathfrak{A}_T$, the past, denotes the σ-algebra generated by

$$\{A: A \cap (T \leqslant t) \in \mathfrak{A}_t \text{ for every } t\}.$$

$\mathfrak{A}_{T-}$ the strict past [1], is the σ-algebra generated by $\mathfrak{A}_t \cap (t < T)$, $t \geqslant 0$.

Let r a real number, define $T_r = \inf\{t > r: X_t \neq X_r\}$. The following example (Example 5.1) due to C. Dolean–Dade, shows that $\mathfrak{A}_r \neq \neq \mathfrak{A}_{T_r}$.

EXAMPLE 5.1. Let $\{B_t\}$ be a Brownian motion relative to a family $\{\mathscr{A}_t\}$ of increasing, right-continuous, containing all null sets. Take $0 < r < s_1 < s_2$, and take $B \in \mathscr{A}_{s_1}$, but $B \neq \mathscr{A}_r$. Let

$$\varphi(s, \omega) = 1_B(\omega)\, 1_{(s_1,s_2]}(s);$$

define

$$X_t = \int_0^t \varphi_s \, dB_s .$$

Clearly $(X_t, \mathscr{A}_t)$ is a continuous martingale. Let $\mathscr{F}_t = \overline{\sigma(X_s: s \leqslant t)}$ be the completed σ-algebra generated by $\{X_t\}$. We may and do assume that $\{\mathscr{F}_t\}$ is right-continuous, for otherwise we replace $\mathscr{F}_t$ by $\bar{\mathscr{F}}_t = \bigcap_{s>t} \mathscr{F}_s$. Take

$$0 < r < s_1 < t < s_2 \qquad \text{and} \qquad A = \{\omega: X_s = X_r \text{ on } [r, t]\}.$$

Here $X_r \equiv 0$. Then $A =$ complement $B \notin \mathscr{A}_r$, so that $A \notin \mathscr{F}_r$. $A \in \mathscr{F}_T$, since

$$\begin{aligned} A \cap \{T_r \leqslant s\} &= \phi && \text{if} \quad s < t, \\ &= A \cap \{T_r \leqslant t\}, && \text{if} \quad s \geqslant t. \end{aligned}$$

$\phi \in \mathscr{F}_T$ (trivially), $A \cap \{T_r \leqslant t] \in \mathscr{F}_t$, for $A \in \mathscr{F}_t$.

As noted T_r takes on infinity as a value. One could modify the example to have T_r finite.

EXAMPLE 5.2. Let

$$\Psi(s, \omega) = 1_B(\omega)\, 1_{(s_1,s_2]}(s) + 1_{(s_3,s_4]}(s),$$

where $0 < r < s_1 < s_2 < s_3 < s_4$. Everything else is as before (Example 5.1), with the resulting T_r is finite. Since $A = \{t < T_r\}$, $A \in \mathscr{F}_{T_{r-}}$ by [3]. So not only $\mathscr{F}_{T_r} \neq \mathscr{F}_r$, but further $(\mathscr{F}_{T-} \subset \mathscr{F}_T)$, $\mathscr{F}_{T_{r-}} \neq \mathscr{F}_r$.

Now we claim that there is a martingale $\{Y_t\}$ defined on $\{X_t\}$, such that $Y_{T_r} \neq Y_r$. Simply take $Y_t = E\{1_A \mid \mathscr{F}_t\}$, where $\mathscr{F}_t = \sigma(X_s: s \leqslant t)$,

$$Y_r = E\{1_A \mid \mathscr{F}_r\}, \qquad \text{and} \qquad E\{Y_t \mid \mathscr{F}_{t \wedge T_r}\} = E\{1_A \mid \mathscr{F}_{t \wedge T_r}\}.$$

By the martingale property the left-hand side is $Y_{t\wedge T_r}$; therefore (by letting $t \to \infty$),

$$Y_T = \lim_{t\to\infty} E\{1_A \mid \mathscr{F}_{t\wedge T_r}\} = 1_A .$$

Since $A \notin \mathscr{F}_r$, $Y_r \neq Y_{T_r}$.

In preparation for a general theorem under which stochastic integral representation is valid, recall that $A_t = \langle X, X\rangle_t$. Let $S(t) = \inf\{r: A_r \geq t\}$ and $T(t) = \inf\{r: A_r > t\}$. S and T are increasing, with $\mathfrak{A}_t$-stopping times for each t. is right-continuous; S is left-continuous. Since $A_t = \inf\{r: T(r) > t\}$, A_t is $\mathfrak{A}_{T(t)}$-stopping time. Denoting by P' the predictable sets generated by

$$g(a)\, 1_{(a,b]}(s),$$

where $g(a)$ is $\mathfrak{A}_{T(a)}$-measurable, we have:

Lemma 5.1. *If $f(s, \omega)$, is P predictable, then $f(T(s), \omega)$ is P' predictable.*

Proof. It is enough (and we will do just that), to prove the lemma for functions of the form

$$\varphi(s, \omega) = g(a)\, 1_{(a,b]}(s),$$

where $g(a)$ is $\mathfrak{A}_a$-measurable. By elementary manipulations,

$$\begin{aligned}\psi(s, \omega) = \varphi(T(s), \omega) &= g(a)\, 1_{(a,b]}(Ts)\\ &= g(a)\, 1_{(A(a),A(b)]}(s),\end{aligned}$$

$g(a)$ is $\mathfrak{A}_{TA(a)}$-measurable. This simply follows from $\mathfrak{A}_{TA(a)} \supset \mathfrak{A}_a$, since $TA(a) \geq a$.

Hence ψ is P'-predictable.

Lemma 5.2. *If $f(s, \omega)$ is P' predictable then $f(A(s), \omega)$ is P-predictable.*

Proof. Again we prove the lemma for functions of the form

$$\varphi(s, \omega) = g(a)\, 1_{(a,b]}(s),$$

where $g(a)$ is $\mathfrak{A}_{T(a)}$-measurable. Now

$$\begin{aligned}\psi(s, \omega) &= \varphi(A(s), \omega)\\ &= g(a)\, 1_{(a,b]}(As)\\ &= g(a)\, 1_{(Ta,Tb]}(s),\end{aligned}$$

since $Ta < s \leq Tb$ if and only if $Ta < TA(s) \leq Tb$. Hence ψ is P predictable.

In order to utilize the so-called random time change, we join to $(\Omega, \mathfrak{A}, P)$ a probability space $(\Omega', \mathfrak{A}', P')$ independent of $(\Omega, \mathfrak{A}, P)$, and carrying a standard Brownian motion $\{B_t'\}$. Let $A_\infty = \lim_{t\to\infty} A_t$; then

$$\hat{B}_t(\omega, \omega') = X(T(t), \omega) + [(B_t'(\omega') - B'_{A_\infty}(\omega')] \cdot 1_{(t>A_\infty)}, \qquad (*)$$

using [5, pp. 292] (see also [1, pp. 122]), is a Brownian motion. This formula suggests (at least), the possibility of representations of representing $Y \in \mathscr{M}$, up to $t < S_\infty = \lim_{t\to\infty} S(t)$. More precisely we have

THEOREM 5.1. *For $Y \in \mathscr{M}$, if $Y(T(t))$ is $\sigma(X(T(s)): s \leqslant t)$-measurable, then* $Y_t = \int_0^t \phi_s \, dX_s$, $R_0 < t < S_\infty$.

Proof. By (*),

$$\begin{aligned} Y(T(t)) &= \int_0^t \phi_s \, dX(T(s)), \qquad t \leqslant A_\infty, \\ &= \int_0^{T(t)} \phi_{As} \, dX_s, \end{aligned}$$

using Lemma 5.2 and the fact that X and A have the same intervals of constancy. Let $[R_1, R_2]$, $[R_3, R_4]$, $\cdots$ be the invervals of constancy of A, $R_1 > R_0 = \inf\{t: X_t \neq 0\}$. These may be identified more precisely, by letting $T_i =$ the time of the ith jump of T, then $R_i = S(T_i)$ and $R_{i+1} = T(T_i)$.

For $t \in (R_0, R_1)$, (R_2, R_3),..., T is one to one; i.e., not only $AT(t) = t$, but also $TA(t) = t$. Consequently

$$\begin{aligned} Y(t) = Y(TA(t)) &= \int_0^{A(t)} \varphi_s \, dX(Ts) \\ &= \int_0^{TA(t)} \varphi_{As} \, dX_s \\ &= \int_0^t \varphi_{A_s} \, dX_s . \end{aligned}$$

The evaluation of $Y(t)$ elsewhere is a little more delicate.

Put $V_t = Y(T(t))$, and note that the T_i's are stopping times relative to $\mathfrak{A}_{T(t)}$. Now:

$$\begin{aligned} V_{T_{i-}} = \lim_{t \nearrow T_i} V_t &= \int_0^{S(Ti)} \varphi_{As} \, dX_s \\ &= \int_0^{R_i} \varphi_{As} \, dX_s , \end{aligned}$$

and

$$V_{T_i} = \int_0^{T(Ti)} \varphi_{As} \, dX_s = \int_0^{R_{i+1}} \varphi_{As} \, dX_s .$$

Again, by constancy of X_s on $[R_i, R_{i+1}]$, $V_{T_{i-}} = V_{T_i}$, implying $Y(R_i) = Y(R_{i+1}) = Y(t)$, $R_i \leqslant t \leqslant R_{i+1}$ by the martingale property of Y, since the R_i's are $\mathfrak{A}_t$-stopping times.

COROLLARY 5.1. *If in addition to the hypothesis of Theorem* 5.2, Y_t *is continuous and constant on* $[S_\infty, \infty)$, *then*

$$Y_t = \int_0^t \psi_s \, dX_s .$$

6. Poisson Process Case

Let $\{X_t\}$ be, now, a right-continuous square integrable martingale. By [11], $X_t = X_t^c + X_t^d$, X^c being the continuous component and X^d the totally discontinuous part. Using [11, 16]:

$$f(X_t) = f(X_0) + \int_0^t f'(X_{s-}) \, dX_s + \tfrac{1}{2} \int_0^t f''(X_{s-}) \, d\langle X^c, X^c\rangle_s$$
$$+ \sum_{s \leqslant t} [f(X_s) - f(X_{s-}) - f'(X_{s-}) \, \Delta X_s]$$

for any twice continuously differentiable f.

Specializing to Poisson process, let $\{N_t\}$ be a Poisson process, with intensity $\lambda = 1$. As is well known $X_t = N_t - t$ is a martingale. For this X, $f(X_t) = \int_0^t f'(X_{s-}) \, dX_s + \sum_{s \leqslant t} [f(X_s) - f(X_{s-}) - f'(X_{s-}) \, \Delta X_s]$. Note that $X^c = 0$ in this case. By applying the formula on the f's of the form $f(x, y, \ldots) = xy \cdots$, and observing that $\Delta X_s = 0$ or 1, we may represent any martingale Y defined on $\{X_t\}$ by the methods of Section 3.

Remarks. Dellacharie [6], uses Levey characterization of Brownian motion in his proof. We have used mathematical induction. Dellacharie's proof was brought to our attention after we wrote Sections 3 and 6.

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